Standardized Test Questions

You may use a graphing calculator to solve the following problems.

35. True or False The domain of $y = \sin^{-1} x$ is $-1 \le x \le 1$. Justify your answer.

36. True or False The domain of $y = \tan^{-1} x$ is $-1 \le x \le 1$. Justify your answer.

37. Multiple Choice Which of the following is $\dfrac{d}{dx}\sin^{-1}\left(\dfrac{x}{2}\right)$?

(A) $-\dfrac{2}{\sqrt{4-x^2}}$ (B) $-\dfrac{1}{\sqrt{4-x^2}}$ (C) $\dfrac{2}{4+x^2}$

(D) $\dfrac{2}{\sqrt{4-x^2}}$ (E) $\dfrac{1}{\sqrt{4-x^2}}$

38. Multiple Choice Which of the following is $\dfrac{d}{dx}\tan^{-1}(3x)$?

(A) $-\dfrac{3}{1+9x^2}$ (B) $-\dfrac{1}{1+9x^2}$ (C) $\dfrac{1}{1+9x^2}$

(D) $\dfrac{3}{1+9x^2}$ (E) $\dfrac{3}{\sqrt{1-9x^2}}$

39. Multiple Choice Which of the following is $\dfrac{d}{dx}\sec^{-1}(x^2)$?

(A) $\dfrac{2}{x\sqrt{x^4-1}}$ (B) $\dfrac{2}{x\sqrt{x^2-1}}$ (C) $\dfrac{2}{x\sqrt{1-x^4}}$

(D) $\dfrac{2}{x\sqrt{1-x^2}}$ (E) $\dfrac{2x}{\sqrt{1-x^4}}$

40. Multiple Choice Which of the following is the slope of the tangent line to $y = \tan^{-1}(2x)$ at $x = 1$?

(A) $-2/5$ (B) $1/5$ (C) $2/5$ (D) $5/2$ (E) 5

Explorations

In Exercises 41–46, find **(a)** the right end behavior model, **(b)** the left end behavior model, and **(c)** any horizontal tangents for the function if they exist.

41. $y = \tan^{-1} x$ **42.** $y = \cot^{-1} x$

43. $y = \sec^{-1} x$ **44.** $y = \csc^{-1} x$

45. $y = \sin^{-1} x$ **46.** $y = \cos^{-1} x$

Extending the Ideas

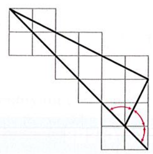
47. *Identities* Confirm the following identities for $x > 0$.

(a) $\cos^{-1} x + \sin^{-1} x = \pi/2$

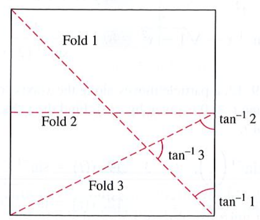
(b) $\tan^{-1} x + \cot^{-1} x = \pi/2$

(c) $\sec^{-1} x + \csc^{-1} x = \pi/2$

48. *Proof Without Words* The figure gives a proof without words that $\tan^{-1} 1 + \tan^{-1} 2 + \tan^{-1} 3 = \pi$. Explain what is going on.



49. *(Continuation of Exercise 48)* Here is a way to construct $\tan^{-1} 1$, $\tan^{-1} 2$, and $\tan^{-1} 3$ by folding a square of paper. Try it and explain what is going on.



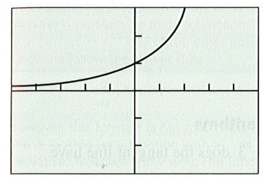
4.4 Derivatives of Exponential and Logarithmic Functions

What you will learn about . . .

- Derivative of e^x
- Derivative of a^x
- Derivative of $\ln x$
- Derivative of $\log_a x$
- Power Rule for Arbitrary Real Powers

and why . . .

Exponential functions are involved in the modeling of growth rates in the real world.



$[-4.9, 4.9]$ by $[-2.9, 2.9]$

(a)

X	Y1
-.03	.98515
-.02	.99007
-.01	.99502
0	ERROR
.01	1.005
.02	1.0101
.03	1.0152

X=0

(b)

Figure 4.16 (a) The graph and (b) the table support the conclusion that

$$\lim_{h\to 0}\frac{e^h - 1}{h} = 1.$$

Derivative of e^x

At the end of the brief review of exponential functions in Section 1.3, we mentioned that the function $y = e^x$ was a particularly important function for modeling exponential growth. The number e was defined in that section to be the limit of $(1 + 1/x)^x$ as $x \to \infty$. This intriguing number shows up in other interesting limits as well, but the one with the most interesting implications for the *calculus* of exponential functions is this one:

$$\lim_{h\to 0}\frac{e^h - 1}{h} = 1.$$

(The graph and the table in Figure 4.16 provide strong support for this limit being 1. A formal algebraic proof that begins with our limit definition of e would require some rather subtle limit arguments, so we will not include one here.)

The fact that the limit is 1 creates a remarkable relationship between the function e^x and its derivative, as we will now see.

$$\begin{aligned}\frac{d}{dx}(e^x) &= \lim_{h\to 0}\frac{e^{x+h} - e^x}{h}\\ &= \lim_{h\to 0}\frac{e^x\cdot e^h - e^x}{h}\\ &= \lim_{h\to 0}\left(e^x\cdot\frac{e^h - 1}{h}\right)\\ &= e^x\cdot\lim_{h\to 0}\left(\frac{e^h - 1}{h}\right)\\ &= e^x\cdot 1\\ &= e^x\end{aligned}$$

In other words, the derivative of this particular function is itself!

$$\frac{d}{dx}(e^x) = e^x$$

If u is a differentiable function of x, then we have

$$\frac{d}{dx}e^u = e^u\frac{du}{dx}.$$

We will make extensive use of this formula when we study exponential growth and decay in Chapter 7.

EXAMPLE 1 Using the Formula

Find dy/dx if $y = e^{(x+x^2)}$.

SOLUTION

Let $u = x + x^2$ and $y = e^u$. Then

$$\frac{dy}{dx} = e^u\frac{du}{dx}, \quad\text{and}\quad \frac{du}{dx} = 1 + 2x.$$

Thus, $\dfrac{dy}{dx} = e^u\dfrac{du}{dx} = e^{(x+x^2)}(1 + 2x)$.

Now Try Exercise 9.

Is any other function its own derivative?

The zero function is also its own derivative, but this hardly seems worth mentioning. (Its value is always 0 and its slope is always 0.) In addition to e^x, however, we can also say that any constant *multiple* of e^x is its own derivative:

$$\frac{d}{dx}(c \cdot e^x) = c \cdot e^x.$$

The next obvious question is whether there are still *other* functions that are their own derivatives, and this time the answer is no. The only functions that satisfy the condition $dy/dx = y$ are functions of the form $y = ke^x$ (and notice that the zero function can be included in this category). We will prove this significant fact in Chapter 7.

Derivative of a^x

What about an exponential function with a base other than e? We will assume that the base is positive and different from 1, since negative numbers to arbitrary real powers are not always real numbers, and $y = 1^x$ is a constant function.

If $a > 0$ and $a \neq 1$, we can use the properties of logarithms to write a^x in terms of e^x. The formula for doing so is

$$a^x = e^{x \ln a}. \quad e^{x \ln a} = e^{\ln(a^x)} = a^x$$

We can then find the derivative of a^x with the Chain Rule.

$$\frac{d}{dx}a^x = \frac{d}{dx}e^{x \ln a} = e^{x \ln a} \cdot \frac{d}{dx}(x \ln a) = e^{x \ln a} \cdot \ln a = a^x \ln a$$

Thus, if u is a differentiable function of x, we get the following rule.

For $a > 0$ and $a \neq 1$,

$$\frac{d}{dx}(a^u) = a^u \ln a \frac{du}{dx}.$$

EXAMPLE 2 Reviewing the Algebra of Logarithms

At what point on the graph of the function $y = 2^t - 3$ does the tangent line have slope 21?

SOLUTION

The slope is the derivative:

$$\frac{d}{dt}(2^t - 3) = 2^t \cdot \ln 2 - 0 = 2^t \ln 2.$$

We want the value of t for which $2^t \ln 2 = 21$. We could use the solver on the calculator, but we will use logarithms for the sake of review.

$$2^t \ln 2 = 21$$

$$2^t = \frac{21}{\ln 2}$$

$$\ln 2^t = \ln\left(\frac{21}{\ln 2}\right) \quad \text{Logarithm of both sides}$$

$$t \cdot \ln 2 = \ln 21 - \ln(\ln 2) \quad \text{Properties of logarithms}$$

$$t = \frac{\ln 21 - \ln(\ln 2)}{\ln 2}$$

$$t \approx 4.921$$

$$y = 2^t - 3 \approx 27.297 \quad \text{Using the stored value of } t$$

The point is approximately $(4.9, 27.3)$. ***Now Try Exercise 29.***

EXPLORATION 1 Leaving Milk on the Counter

A glass of cold milk from the refrigerator is left on the counter on a warm summer day. Its temperature y (in degrees Fahrenheit) after sitting on the counter t minutes is

$$y = 72 - 30(0.98)^t.$$

Answer the following questions by interpreting y and dy/dt.

1. What is the temperature of the refrigerator? How can you tell?
2. What is the temperature of the room? How can you tell?
3. When is the milk warming up the fastest? How can you tell?
4. Determine algebraically when the temperature of the milk reaches 55°F.
5. At what rate is the milk warming when its temperature is 55°F? Answer with an appropriate unit of measure.

This equation answers what was once a perplexing problem: Is there a function with derivative x^{-1}? All of the other power functions follow the Power Rule,

$$\frac{d}{dx}x^n = nx^{n-1}.$$

However, this formula is not much help if one is looking for a function with x^{-1} as its derivative! Now we know why: The function we should be looking for is not a power function at all; it is the natural logarithm function.

Derivative of $\ln x$

Now that we know the derivative of e^x, it is relatively easy to find the derivative of its inverse function, $\ln x$.

$$y = \ln x$$

$$e^y = x \quad \text{Inverse function relationship}$$

$$\frac{d}{dx}(e^y) = \frac{d}{dx}(x) \quad \text{Differentiate implicitly.}$$

$$e^y \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{x}$$

If u is a differentiable function of x and $u > 0$,

$$\frac{d}{dx}\ln u = \frac{1}{u}\frac{du}{dx}.$$

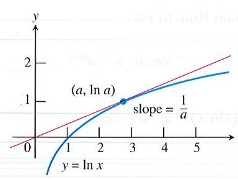


Figure 4.17 The tangent line intersects the curve at some point $(a, \ln a)$, where the slope of the curve is $1/a$. (Example 3)

EXAMPLE 3 A Tangent Through the Origin

A line with slope m passes through the origin and is tangent to the graph of $y = \ln x$. What is the value of m?

SOLUTION

This problem is a little harder than it looks, since we do not know the point of tangency. However, we do know two important facts about that point:

1. it has coordinates $(a, \ln a)$ for some positive a, and
2. the tangent line there has slope $m = 1/a$ (Figure 4.17).

Since the tangent line passes through the origin, its slope is

$$m = \frac{\ln a - 0}{a - 0} = \frac{\ln a}{a}.$$

continued

Setting these two formulas for m equal to each other, we have

$$\frac{\ln a}{a} = \frac{1}{a}$$
$$\ln a = 1$$
$$e^{\ln a} = e^1$$
$$a = e$$
$$m = \frac{1}{e}.$$

Now Try Exercise 31.

Derivative of $\log_a x$

To find the derivative of $\log_a x$ for an arbitrary base ($a > 0, a \neq 1$), we use the change-of-base formula for logarithms to express $\log_a x$ in terms of natural logarithms, as follows:

$$\log_a x = \frac{\ln x}{\ln a}.$$

The rest is easy:

$$\begin{aligned}\frac{d}{dx}\log_a x &= \frac{d}{dx}\left(\frac{\ln x}{\ln a}\right)\\ &= \frac{1}{\ln a}\cdot\frac{d}{dx}\ln x \qquad \text{Since } \ln a \text{ is a constant}\\ &= \frac{1}{\ln a}\cdot\frac{1}{x}\\ &= \frac{1}{x\ln a}.\end{aligned}$$

So, if u is a differentiable function of x and $u > 0$, the formula is as follows.

For $a > 0$ and $a \neq 1$,

$$\frac{d}{dx}\log_a u = \frac{1}{u\ln a}\frac{du}{dx}.$$

EXAMPLE 4 Going the Long Way with the Chain Rule

Find dy/dx if $y = \log_a a^{\sin x}$.

SOLUTION

Carefully working from the outside in, we apply the Chain Rule to get:

$$\begin{aligned}\frac{d}{dx}(\log_a a^{\sin x}) &= \frac{1}{a^{\sin x}\ln a}\cdot\frac{d}{dx}(a^{\sin x}) \qquad \log_a u,\ u = a^{\sin x}\\ &= \frac{1}{a^{\sin x}\ln a}\cdot a^{\sin x}\ln a\cdot\frac{d}{dx}(\sin x) \qquad a^u,\ u = \sin x\\ &= \frac{a^{\sin x}\ln a}{a^{\sin x}\ln a}\cdot\cos x\\ &= \cos x.\end{aligned}$$

Now Try Exercise 23.

We could have saved ourselves a lot of work in Example 4 if we had noticed at the beginning that $\log_a a^{\sin x}$, being the composite of inverse functions, is equal to $\sin x$. It is always a good idea to simplify functions *before* differentiating, wherever possible. On the other hand, it is comforting to know that all these rules do work if applied correctly.

Power Rule for Arbitrary Real Powers

We are now ready to prove the Power Rule in its final form. As long as $x > 0$, we can write any real power of x as a power of e, specifically

$$x^n = e^{n\ln x}.$$

This enables us to differentiate x^n for any real power n, as follows:

$$\begin{aligned}\frac{d}{dx}(x^n) &= \frac{d}{dx}(e^{n\ln x})\\ &= e^{n\ln x}\cdot\frac{d}{dx}(n\ln x) \qquad e^u,\ u = n\ln x\\ &= e^{n\ln x}\cdot\frac{n}{x}\\ &= x^n\cdot\frac{n}{x}\\ &= nx^{n-1}.\end{aligned}$$

The Chain Rule extends this result to the Power Rule's final form.

RULE 10 Power Rule for Arbitrary Real Powers

If u is a positive differentiable function of x and n is any real number, then u^n is a differentiable function of x, and

$$\frac{d}{dx}u^n = nu^{n-1}\frac{du}{dx}.$$

EXAMPLE 5 Using the Power Rule in All Its Power

(a) If $y = x^{\sqrt{2}}$, then

$$\frac{dy}{dx} = \sqrt{2}x^{(\sqrt{2}-1)}.$$

(b) If $y = (2 + \sin 3x)^{\pi}$, then

$$\begin{aligned}\frac{d}{dx}(2+\sin 3x)^{\pi} &= \pi(2+\sin 3x)^{\pi-1}(\cos 3x)\cdot 3\\ &= 3\pi(2+\sin 3x)^{\pi-1}(\cos 3x).\end{aligned}$$

Now Try Exercise 35.

EXAMPLE 6 Finding Domain

If $f(x) = \ln(x-3)$, find $f'(x)$. State the domain of f'.

SOLUTION

The domain of f is $(3, \infty)$ and

$$f'(x) = \frac{1}{x-3}.$$

continued

The domain of f' appears to be all $x \neq 3$. However, since f is not defined for $x < 3$, neither is f'. Thus,

$$f'(x) = \frac{1}{x-3}, \quad x > 3.$$

That is, the domain of f' is $(3, \infty)$. **Now Try Exercise 37.**

Sometimes the properties of logarithms can be used to simplify the differentiation process, even if we must introduce the logarithms ourselves as a step in the process. Example 7 shows a clever way to differentiate $y = x^x$ for $x > 0$.

EXAMPLE 7 Logarithmic Differentiation

Find dy/dx for $y = x^x$, $x > 0$.

SOLUTION

$$y = x^x$$

$$\ln y = \ln x^x \qquad \text{Logs of both sides}$$

$$\ln y = x \ln x \qquad \text{Property of logs}$$

$$\frac{d}{dx}(\ln y) = \frac{d}{dx}(x \ln x) \qquad \text{Differentiate implicitly.}$$

$$\frac{1}{y}\frac{dy}{dx} = 1 \cdot \ln x + x \cdot \frac{1}{x}$$

$$\frac{dy}{dx} = y(\ln x + 1)$$

$$\frac{dy}{dx} = x^x(\ln x + 1)$$

Now Try Exercise 43.

EXAMPLE 8 How Fast Does a Flu Spread?

The spread of a flu in a certain school is modeled by the equation

$$P(t) = \frac{100}{1 + e^{3-t}},$$

where $P(t)$ is the total number of students infected t days after the flu was first noticed. Many of them may already be well again at time t.

(a) Estimate the initial number of students infected with the flu.

(b) How fast is the flu spreading after 3 days?

(c) When will the flu spread at its maximum rate? What is this rate?

SOLUTION

The graph of P as a function of t is shown in Figure 4.18.

(a) $P(0) = 100/(1 + e^3) = 5$ students (to the nearest whole number).

continued

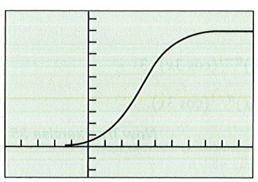


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Figure 4.18 The graph of

$$P(t) = \frac{100}{1 + e^{3-t}},$$

modeling the spread of a flu. (Example 8)

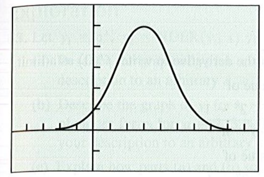

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Figure 4.19 The graph of dP/dt, the rate of spread of the flu in Example 8. The graph of P is shown in Figure 4.18.

(b) To find the rate at which the flu spreads, we find dP/dt. To find dP/dt, we need to invoke the Chain Rule twice:

$$\frac{dP}{dt} = \frac{d}{dt}(100(1 + e^{3-t})^{-1}) = 100 \cdot (-1)(1 + e^{3-t})^{-2} \cdot \frac{d}{dt}(1 + e^{3-t})$$

$$= -100(1 + e^{3-t})^{-2} \cdot (0 + e^{3-t} \cdot \frac{d}{dt}(3 - t))$$

$$= -100(1 + e^{3-t})^{-2}(e^{3-t} \cdot (-1))$$

$$= \frac{100e^{3-t}}{(1 + e^{3-t})^2}.$$

At $t = 3$, then, $dP/dt = 100/4 = 25$. The flu is spreading to 25 students per day.

(c) We could estimate when the flu is spreading the fastest by seeing where the graph of $y = P(t)$ has the steepest upward slope, but we can answer both the "when" and the "what" parts of this question most easily by finding the maximum point on the graph of the derivative (Figure 4.19).

We see by tracing on the curve that the maximum rate occurs at about 3 days, when (as we have just calculated) the flu is spreading at a rate of 25 students per day. **Now Try Exercise 51.**

Quick Review 4.4 (For help, go to Sections 1.3 and 1.5.)

Exercise numbers with a gray background indicate problems that the authors have designed to be solved *without a calculator.*

1. Write $\log_5 8$ in terms of natural logarithms.

2. Write 7^x as a power of e.

In Exercises 3–7, simplify the expression using properties of exponents and logarithms.

3. $\ln(e^{\tan x})$

4. $\ln(x^2 - 4) - \ln(x + 2)$

5. $\log_2(8^{x-5})$

6. $(\log_4 x^{15})/(\log_4 x^{12})$

7. $3 \ln x - \ln 3x + \ln(12x^2)$

In Exercises 8–10, solve the equation algebraically using logarithms. Give an *exact* answer, such as $(\ln 2)/3$, and also an approximate answer to the nearest hundredth.

8. $3^x = 19$

9. $5^t \ln 5 = 18$

10. $3^{x+1} = 2^x$

Section 4.4 Exercises

In Exercises 1–28, find dy/dx.

1. $y = 2e^x$

2. $y = e^{2x}$

3. $y = e^{-x}$

4. $y = e^{-5x}$

5. $y = e^{2x/3}$

6. $y = e^{-x/4}$

7. $y = xe^2 - e^x$

8. $y = x^2e^x - xe^x$

9. $y = e^{\sqrt{x}}$

10. $y = e^{(x^2)}$

11. $y = 8^x$

12. $y = 9^{-x}$

13. $y = 3^{\csc x}$

14. $y = 3^{\cot x}$

15. $y = \ln(x^2)$

16. $y = (\ln x)^2$

17. $y = \ln(1/x)$

18. $y = \ln(10/x)$

19. $y = \ln(\ln x)$

20. $y = x \ln x - x$

21. $y = \log_4 x^2$

22. $y = \log_5 \sqrt{x}$

23. $y = \log_2(1/x)$

24. $y = 1/\log_2 x$

25. $y = \ln 2 \cdot \log_2 x$

26. $y = \log_3(1 + x \ln 3)$

27. $y = \log_{10} e^x$

28. $y = \ln 10^x$

29. At what point on the graph of $y = 3^x + 1$ is the tangent line parallel to the line $y = 5x - 1$?

30. At what point on the graph of $y = 2e^x - 1$ is the tangent line perpendicular to the line $y = -3x + 2$?

31. A line with slope m passes through the origin and is tangent to $y = \ln(2x)$. What is the value of m?

32. A line with slope m passes through the origin and is tangent to $y = \ln(x/3)$. What is the value of m?

In Exercises 33–36, find dy/dx.

33. $y = x^{\pi}$

34. $y = x^{1+\sqrt{2}}$

35. $y = x^{-\sqrt{2}}$

36. $y = x^{1-e}$

In Exercises 37–42, find $f'(x)$ and state the domain of f'.

37. $f(x) = \ln(x + 2)$

38. $f(x) = \ln(2x + 2)$

39. $f(x) = \ln(2 - \cos x)$

40. $f(x) = \ln(x^2 + 1)$

41. $f(x) = \log_2(3x + 1)$

42. $f(x) = \log_{10}\sqrt{x + 1}$

Group Activity In Exercises 43–48, use the technique of logarithmic differentiation to find dy/dx.

43. $y = (\sin x)^x, \quad 0 < x < \pi/2$

44. $y = x^{\tan x}, \quad x > 0$

45. $y = \sqrt[5]{\dfrac{(x - 3)^4(x^2 + 1)}{(2x + 5)^3}}$

46. $y = \dfrac{x\sqrt{x^2 + 1}}{(x + 1)^{2/3}}$

47. $y = x^{\ln x}$ 48. $y = x^{(1/\ln x)}$

49. Find an equation for a line that is tangent to the graph of $y = e^x$ and goes through the origin.

50. Find an equation for a line that is normal to the graph of $y = xe^x$ and goes through the origin.

51. ***Spread of a Rumor*** The spread of a rumor in a certain school is modeled by the equation

$$P(t) = \frac{300}{1 + 2^{4-t}},$$

where $P(t)$ is the total number of students who have heard the rumor t days after the rumor first started to spread.

(a) Estimate the initial number of students who first heard the rumor.

(b) How fast is the rumor spreading after 4 days?

(c) When will the rumor spread at its maximum rate? What is that rate?

52. ***Spread of Flu*** The spread of flu in a certain school is modeled by the equation

$$P(t) = \frac{200}{1 + e^{5-t}},$$

where $P(t)$ is the total number of students infected t days after the flu first started to spread.

(a) Estimate the initial number of students infected with this flu.

(b) How fast is the flu spreading after 4 days?

(c) When will the flu spread at its maximum rate? What is that rate?

53. ***Radioactive Decay*** The amount A (in grams) of radioactive plutonium remaining in a 20-gram sample after t days is given by the formula

$$A = 20 \cdot (1/2)^{t/140}.$$

At what rate is the plutonium decaying when $t = 2$ days? Answer in appropriate units.

54. For any positive constant k, the derivative of $\ln(kx)$ is $1/x$. Prove this fact

(a) by using the Chain Rule.

(b) by using a property of logarithms and differentiating.

55. Let $f(x) = 2^x$.

(a) Find $f'(0)$.

(b) Use the definition of the derivative to write $f'(0)$ as a limit.

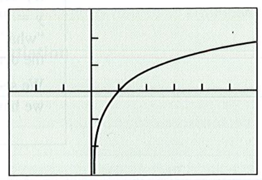
(c) Deduce the exact value of

$$\lim_{h\to 0} \frac{2^h - 1}{h}.$$

(d) What is the exact value of

$$\lim_{h\to 0} \frac{7^h - 1}{h}?$$

56. **Writing to Learn** The graph of $y = \ln x$ looks as though it might be approaching a horizontal asymptote. Write an argument based on the graph of $y = e^x$ to explain why it does not.



[–3, 6] by [–3, 3]

Standardized Test Questions

57. **True or False** The derivative of $y = 2^x$ is 2^x. Justify your answer.

58. **True or False** The derivative of $y = e^{2x}$ is $2(\ln 2)e^{2x}$. Justify your answer.

59. **Multiple Choice** If a flu is spreading at the rate of

$$P(t) = \frac{150}{1 + e^{4-t}},$$

which of the following is the initial number of persons infected?

(A) 1 (B) 3 (C) 7 (D) 8 (E) 75

60. **Multiple Choice** Which of the following is the domain of $f'(x)$ if $f(x) = \log_2(x + 3)$?

(A) $x < -3$ (B) $x \le 3$ (C) $x \ne -3$
(D) $x > -3$ (E) $x \ge -3$

61. **Multiple Choice** Which of the following gives dy/dx if $y = \log_{10}(2x - 3)$?

(A) $\dfrac{2}{(2x - 3)\ln 10}$ (B) $\dfrac{2}{2x - 3}$ (C) $\dfrac{1}{(2x - 3)\ln 10}$
(D) $\dfrac{1}{2x - 3}$ (E) $\dfrac{1}{2x}$

62. **Multiple Choice** Which of the following gives the slope of the tangent line to the graph of $y = 2^{1-x}$ at $x = 2$?

(A) $-\dfrac{1}{2}$ (B) $\dfrac{1}{2}$ (C) -2 (D) 2 (E) $-\dfrac{\ln 2}{2}$

Exploration

63. Let $y_1 = a^x$, $y_2 = \text{NDER}(y_1, x)$ $y_3 = y_2/y_1$, and $y_4 = e^{y_3}$.

(a) Describe the graph of y_4 for $a = 2, 3, 4, 5$. Generalize your description to an arbitrary $a > 1$.

(b) Describe the graph of y_3 for $a = 2, 3, 4, 5$. Compare a table of values for y_3 for $a = 2, 3, 4, 5$ with $\ln a$. Generalize your description to an arbitrary $a > 1$.

(c) Explain how parts (a) and (b) support the statement

$$\frac{d}{dx}a^x = a^x \quad \text{if and only if} \quad a = e.$$

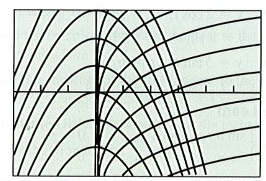
(d) Show algebraically that $y_1 = y_2$ if and only if $a = e$.

Extending the Ideas

64. ***Orthogonal Families of Curves*** Prove that all curves in the family

$$y = -\frac{1}{2}x^2 + k$$

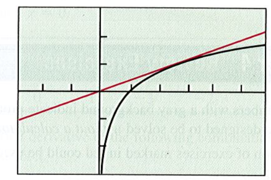
(k any constant) are perpendicular to all curves in the family $y = \ln x + c$ (c any constant) at their points of intersection. (See accompanying figure.)



[–3, 6] by [–3, 3]

65. ***Which is Bigger, π^e or e^π?*** Calculators have taken some of the mystery out of this once-challenging question. (Go ahead and check; you will see that it is a surprisingly close call.) You can answer the question without a calculator, though, by using the result from Example 3 of this section.

Recall from that example that the line through the origin tangent to the graph of $y = \ln x$ has slope $1/e$.



[–3, 6] by [–3, 3]

(a) Find an equation for this tangent line.

(b) Give an argument based on the graphs of $y = \ln x$ and the tangent line to explain why $\ln x < x/e$ for all positive $x \ne e$.

(c) Show that $\ln(x^e) < x$ for all positive $x \ne e$.

(d) Conclude that $x^e < e^x$ for all positive $x \ne e$.

(e) So which is bigger, π^e or e^π?

Quick Quiz for AP* Preparation: Sections 4.3–4.4

1. **Multiple Choice** If $f(x) = \ln(x + 4 + e^{-3x})$, then $f'(0)$ is

(A) $-\dfrac{2}{5}$ (B) $\dfrac{1}{5}$ (C) $\dfrac{1}{4}$ (D) $\dfrac{2}{5}$ (E) nonexistent

2. **Multiple Choice** Let f be the function defined by $f(x) = x^3 + x$. If $g(x) = f^{-1}(x)$ and $g(2) = 1$, what is the value of $g'(2)$?

(A) $\dfrac{1}{13}$ (B) $\dfrac{1}{4}$ (C) $\dfrac{7}{4}$ (D) 4 (E) 13

3. **Multiple Choice** Which of the following gives $\dfrac{dy}{dx}$ if $y = \sin^{-1}(2x)$?

(A) $-\dfrac{2}{\sqrt{1 - 4x^2}}$ (B) $-\dfrac{1}{\sqrt{1 - 4x^2}}$ (C) $\dfrac{2}{\sqrt{1 - 4x^2}}$
(D) $\dfrac{1}{\sqrt{1 - 4x^2}}$ (E) $\dfrac{2x}{1 + 4x^2}$

4. **Free Response** A particle moves along the x-axis with position at time t given by $x(t) = e^t \sin t$ for $0 \le t \le 2\pi$.

(a) Find each time t, $0 \le t \le 2\pi$, for which the particle is at rest.

(b) Find the value of the constant A for which $Ax''(t) + x'(t) = x(t)$ for $0 \le t \le 2\pi$.